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1989 J. Phys. A: Math. Gen. 22 3723

(http://iopscience.iop.org/0305-4470/22/17/037)

J. Phys. A: Math. Gen. 22 (1989) 3723–3730. Printed in the UK

COMMENT

An sl(4, ℝ) Lie algebraic approach to the Bargmann functions and its application to the second Pöschl–Teller equation

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Received 20 December 1988

Abstract. The sl(4, ℝ) Lie algebraic treatment of the Wigner SU(2) matrices is extended by analytic continuation to the Bargmann SU(1, 1) matrices corresponding to the positive discrete series irreducible representations. It is then used to obtain an sl(4, ℝ) dynamical potential algebra for the negative-energy solutions of the second Pöschl–Teller equation.

In a recent paper (Quesne 1988), it was shown that the sl(4, ℝ) Lie algebra has a useful application to the first Pöschl–Teller equation (Pöschl and Teller 1933). Its generators can indeed connect together both solutions of the latter corresponding to the same potential strength but to different energies, and solutions with the same energy but different quantised potential strengths: sl(4, ℝ) is a so-called dynamical potential algebra for the first family of Pöschl–Teller potentials.

Whether sl(4, ℝ) can play the same role for other exactly solvable one-dimensional potentials is an interesting question, whereon we will comment in the present paper.

In the study of the first Pöschl–Teller equation, the physically relevant sl(4, ℝ) subalgebra was the maximal compact one, so(4). Due to the discrete nature of the spectrum, the so(4) generators indeed connect together all the solutions with the same energy, but different potential strengths, and is therefore a potential algebra for the first family of Pöschl–Teller potentials (Barut *et al* 1987a). Hence the known transformation properties of the Wigner SU(2) matrices (Wigner 1959) under so(4) and sl(4, ℝ) could be used to obtain those of the Pöschl–Teller equation solutions, to which they are related.

Such simplifications do not occur for other one-dimensional potentials, whose spectrum contains a continuum of positive energy levels in addition to a finite number of negative eigenvalues. The role of potential algebra is then played by so(2, 2) instead of so(4) (Frank and Wolf 1985, Barut *et al* 1987b), and the solutions of the equation are related to the Bargmann SU(1, 1) matrices (Bargmann 1947) instead of the Wigner SU(2) matrices. To the author's knowledge, the behaviour of Bargmann functions under sl(4, ℝ) ⊃ so(2, 2) has not been studied so far.

In this comment, we shall fill in this gap for those Bargmann functions corresponding to the SU(1, 1) positive discrete series irreducible representations (irreps), and we shall then apply the results to the negative-energy solutions of the second Pöschl–Teller equation (Pöschl and Teller 1933). Our procedure is based on the known properties of the Wigner SU(2) matrices under analytic continuation in the plane of complex angular momentum (Holman and Biedenharn 1966, Ui 1968, 1970).

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0305-4470/89/173723+08$02.50

Let us first briefly review the behaviour of the (complex conjugate) Wigner SU(2) matrices

$$D^{j*}_{m'm}(\alpha, \beta, \gamma) = \exp(\mathrm{i}m'\alpha) d^{j}_{m'm}(\beta) \exp(\mathrm{i}m\gamma) \tag{1}$$

under $\mathrm{sl}(4, \mathbb{R}) \supset \mathrm{so}(4)$ (Quesne 1988). Here α, β, γ ($0 \leqslant \alpha, \gamma < 2\pi, 0 \leqslant \beta \leqslant \pi$) represent Euler angles, j runs over all integers and half-integers, and m' and m over $-j, -j+1, \ldots, j$. With respect to the $\mathrm{so}(4) \simeq \mathrm{su}(2) \oplus \mathrm{su}(2)$ algebra, generated by two commuting sets of angular momentum operators† J_0, J_+, J_-, and I_0, I_+, I_-, the $(2j+1)^2$ functions $D^{j*}_{m'm}$ corresponding to a given j value, form a basis of an irrep labelled by $[N0] \simeq (j, j)$, where $N = 2j$. The Casimir operators $\boldsymbol{J}^2$ and $\boldsymbol{I}^2$ of both su(2) subalgebras coincide, and their common eigenvalues are equal to $j(j+1)$. The action of the operators $J_0, J_\pm$, and $I_0, I_\pm$ on the complex conjugate Wigner functions is that of standard angular momentum operators on states $|jm'\rangle$ and $|jm\rangle$ respectively.

By adding to the operators $J_0, J_\pm, I_0$, and $I_\pm$, the nine components $U_{\sigma\tau}$, $\sigma, \tau = +1, 0, -1$, of an irreducible tensor of rank (1, 1) with respect to $\mathrm{su}(2) \oplus \mathrm{su}(2)$, one obtains the generators of an $\mathrm{sl}(4, \mathbb{R})$ algebra. With respect to the latter, the set of complex conjugate Wigner functions separates into two subsets, corresponding to all integral or half-integral values of j, respectively. Both carry an $\mathrm{sl}(4, \mathbb{R})$ unitary irrep of the ladder series $\mathfrak{D}^{\mathrm{ladd}}(j_0, j_0; \eta)$, characterised by the real parameter η appearing in the definition of $U_{\sigma\tau}$, and by the minimum j value j_0, equal to 0 or $\frac{1}{2}$, depending on whether j is integral or half-integral. For such irreps, all three $\mathrm{sl}(4, \mathbb{R})$ independent Casimir operators assume unique numerical (η-dependent) values. The action of $U_{\sigma\tau}$ on the complex conjugate Wigner functions results from a straightforward application of the Wigner–Eckart theorem with respect to $\mathrm{su}(2) \oplus \mathrm{su}(2)$, and is given by

$$U_{\sigma\tau} D^{j*}_{m'm}(\alpha, \beta, \gamma) = \sum_{j'} a_{j',j} \langle jm', 1\sigma | j'm'+\sigma \rangle \langle jm, 1\tau | j'm+\tau \rangle D^{j'*}_{m'+\sigma, m+\tau}(\alpha, \beta, \gamma) \tag{2}$$

where $\langle jm', 1\sigma | j'm'+\sigma \rangle$ denotes an SU(2) Wigner coefficient, the summation runs over $j' = j-1, j, j+1$, and

$$a_{j+1,j} = \mathrm{i}(j+1) - \tfrac{1}{4}\eta \qquad a_{j,j} = -\tfrac{1}{4}\eta \qquad a_{j-1,j} = -\mathrm{i}j - \tfrac{1}{4}\eta. \tag{3}$$

Let us next consider the (complex conjugate) Bargmann SU(1, 1) matrices corresponding to the positive discrete series irreps D^+_k, $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$,

$$V^{k*}_{m'm}(\alpha, \beta, \gamma) = \exp(\mathrm{i}m'\alpha) v^{k}_{m'm}(\beta) \exp(\mathrm{i}m\gamma) \tag{4}$$

where

$$v^k_{m'm}(\beta) = \begin{cases} \dfrac{1}{(m'-m)!} \left(\dfrac{(m'-k)!(m'+k-1)!}{(m-k)!(m+k-1)!} \right)^{1/2} (\sinh \tfrac{1}{2}\beta)^{m'-m} (\cosh \tfrac{1}{2}\beta)^{-m'-m} & \\ \times {}_2F_1(k-m, 1-m-k; m'-m+1; -\sinh^2 \tfrac{1}{2}\beta) & \text{if } m' \geqslant m \\ (-1)^{m'-m} v^k_{mm'}(\beta) & \text{if } m' < m. \end{cases} \tag{5}$$

Here the variables α, β, γ vary in the intervals $0 \leqslant \alpha, \gamma < 2\pi$, and $0 \leqslant \beta < \infty$, while m' and m run over $k, k+1, \ldots$.

The Bargmann functions $V^{k*}_{m'm}(\alpha, \beta, \gamma)$ can be obtained by an analytic continuation of $D^{j*}_{m'm}(\alpha, \mathrm{i}\beta, \gamma)$ from positive to negative real values of j, and the substitution of k

† The $\mathrm{sl}(4, \mathbb{R})$ generators used in the present comment differ from the generators $J_0, J_\pm, K_0, K_\pm, T_{\sigma\tau}$, of Quesne (1988) by the following algebra automorphism: $J_0 = J_0$, $J_\pm = J_\pm$, $I_0 = -K_0$, $I_\pm = -K_\mp$, and $U_{\sigma\tau} = (-1)^{1+\tau} T_{\sigma,-\tau}$.

for $-j$ (Holman and Biedenharn 1966). In doing so, the functions $D^{j*}_{m'm}(\alpha, \mathrm{i}\beta, \gamma)$ and $V^{k*}_{m'm}(\alpha, \beta, \gamma)$ are considered to be defined for all integral or half-integral values of m' and m, but of course they are found to vanish identically for $|m'|$, $|m|>j$, and m', $m<k$, respectively.

We remark here that the same procedure applied to $D^{0*}_{00}(\alpha, \mathrm{i}\beta, \gamma)$ does not lead to a positive discrete series irrep of SU(1, 1), but to the identity representation, which is its only finite-dimensional unitary irrep. Both the identity representation and the representation $V^{1/2*}_{m'm}$ are not square integrable, as opposed to the representations $V^{k*}_{m'm}$ with $k \geqslant 1$, which satisfy the orthogonality relation

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\gamma \int_0^{+\infty} \mathrm{d}\beta \sinh \beta V^{k'*}_{\mu'\mu}(\alpha, \beta, \gamma) V^{k}_{m'm}(\alpha, \beta, \gamma) = \frac{8\pi^2}{2k-1} \delta_{k',k}\delta_{\mu',m'}\delta_{\mu,m} \qquad k, k' \geqslant 1. \tag{6}$$

When replacing β by $\mathrm{i}\beta$ in the so(4) and sl(4, ℝ) generators, we obtain operators with the same commutation relations, but different Hermiticity properties. By changing the phase of the operators, we can then adjust such properties so as to conform to standard rules.

Denoting by primed operators the sl(4, ℝ) generators wherein the substitution $\beta \to \mathrm{i}\beta$ has been carried out, let us set

$$\mathscr{J}_0 = J'_0 \qquad \mathscr{J}_\pm = -\mathrm{i}J'_\pm \qquad \mathscr{I}_0 = I'_0 \qquad \mathscr{I}_\pm = -\mathrm{i}I'_\pm. \tag{7}$$

From the su(2) commutation relations, we obtain

$$[\mathscr{J}_0, \mathscr{J}_\pm] = \pm\mathscr{J}_\pm \qquad [\mathscr{J}_+, \mathscr{J}_-] = -2\mathscr{J}_0 \tag{8}$$

and similar relations for $\mathscr{I}_0$ and $\mathscr{I}_\pm$. On the other hand, from the explicit expressions of J_0, $J_\pm$, I_0, $I_\pm$ (Quesne 1988), we get

$$\begin{aligned} &\mathscr{J}_0 = -\mathrm{i}\partial_\alpha \qquad && \mathscr{J}_\pm = \mathrm{e}^{\pm\mathrm{i}\alpha}(-\mathrm{i}\coth\beta\, \partial_\alpha \mp \partial_\beta + \mathrm{i}\,\mathrm{cosech}\beta\, \partial_\gamma) \\ &\mathscr{I}_0 = -\mathrm{i}\partial_\gamma && \mathscr{I}_\pm = \mathrm{e}^{\pm\mathrm{i}\gamma}(-\mathrm{i}\,\mathrm{cosech}\beta\, \partial_\alpha \pm \partial_\beta + \mathrm{i}\coth\beta\, \partial_\gamma). \end{aligned} \tag{9}$$

From (9), it follows that

$$(\mathscr{J}_0)^\dagger = \mathscr{J}_0 \qquad (\mathscr{J}_\pm)^\dagger = \mathscr{J}_\mp \tag{10}$$

and similar relations for $\mathscr{I}_0$ and $\mathscr{I}_\pm$, with respect to the measure $\sinh\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma$, used in defining the orthogonality properties (6) of Bargmann functions. Hence, $\mathscr{J}_0$, $\mathscr{J}_\pm$, $\mathscr{I}_0$ and $\mathscr{I}_\pm$ generate an so(2, 2) ≃ su(1, 1)⊕su(1, 1) algebra. The Casimir operators $\boldsymbol{\mathscr{J}}^2$ and $\boldsymbol{\mathscr{I}}^2$ of both su(1, 1) subalgebras again coincide and are given by

$$\begin{aligned} \boldsymbol{\mathscr{J}}^2 &= -\mathscr{J}_+\mathscr{J}_- + \mathscr{J}_0^2 - \mathscr{J}_0 = \boldsymbol{\mathscr{I}}^2 = -\mathscr{I}_+\mathscr{I}_- + \mathscr{I}_0^2 - \mathscr{I}_0 \\ &= \partial^2_{\beta\beta} + \coth\beta\, \partial_\beta + \mathrm{cosech}^2\beta(\partial^2_{\alpha\alpha} - 2\cosh\beta\, \partial^2_{\alpha\gamma} + \partial^2_{\gamma\gamma}). \end{aligned} \tag{11}$$

With respect to su(1, 1)⊕su(1, 1), the Bargmann functions (4) transform under $D^+_k \oplus D^+_k$, and they satisfy the relations

$$\begin{aligned} &\boldsymbol{\mathscr{J}}^2 V^{k*}_{m'm}(\alpha, \beta, \gamma) = k(k-1) V^{k*}_{m'm}(\alpha, \beta, \gamma) \\ &\mathscr{J}_0 V^{k*}_{m'm}(\alpha, \beta, \gamma) = m' V^{k*}_{m'm}(\alpha, \beta, \gamma) \\ &\mathscr{J}_\pm V^{k*}_{m'm}(\alpha, \beta, \gamma) = [(m' \mp k \pm 1)(m' \pm k)]^{1/2} V^{k*}_{m'\pm1,m}(\alpha, \beta, \gamma) \end{aligned} \tag{12}$$

as well as similar equations for $\mathscr{I}_0$ and $\mathscr{I}_\pm$ with m' replaced by m. All such relations can be directly obtained from the known properties of the Wigner matrices by the above-mentioned analytic continuation.

Since the substitution $\beta \to \mathrm{i}\beta$ transforms the Wigner SU(2) matrices into the finite-dimensional non-unitary irrep matrices of SU(1, 1) (Holman and Biedenharn 1966, Ui 1968, 1970), it is obvious that such a replacement in the su(2)⊕su(2) rank (1, 1) irreducible tensor $U_{\sigma\tau}$ should lead to an su(1, 1)⊕su(1, 1) irreducible tensor $\mathscr{U}_{\sigma\tau}$, transforming under the non-unitary irrep (1, 1). By definition (Ui 1968), this irreducible tensor must satisfy the commutation relations

$$[\mathscr{I}_0, \mathscr{U}_{\sigma\tau}] = \sigma\mathscr{U}_{\sigma\tau} \qquad [\mathscr{I}_\pm, \mathscr{U}_{\sigma\tau}] = \mp[(1\mp\sigma)(2\pm\sigma)]^{1/2}\mathscr{U}_{\sigma\pm1,\tau} \tag{13}$$

and similar relations for $\mathscr{I}_0$ and $\mathscr{I}_\pm$ with the role of σ and τ interchanged. From (7) and the defining relations of the su(2)⊕su(2) irreducible tensor $U_{\sigma\tau}$, it results that

$$\mathscr{U}_{\sigma\tau} = \mathrm{i}^{\sigma+\tau} U'_{\sigma\tau} \tag{14}$$

satisfies (13) and its counterpart for $\mathscr{I}_0$ and $\mathscr{I}_\pm$.

From the explicit expressions of $U_{\sigma\tau}$ (Quesne 1988), we obtain

$$\begin{aligned}
\mathscr{U}_{\pm1,\pm1} &= \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha+\gamma)}[\mp\partial_\alpha - \mathrm{i}\sinh\beta\,\partial_\beta \mp \partial_\gamma - (\mathrm{i}-\tfrac{1}{4}\eta)(1+\cosh\beta)]\\
\mathscr{U}_{\pm1,\mp1} &= \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha-\gamma)}[\pm\partial_\alpha - \mathrm{i}\sinh\beta\,\partial_\beta \mp \partial_\gamma + (\mathrm{i}-\tfrac{1}{4}\eta)(1-\cosh\beta)]\\
\mathscr{U}_{\pm1,0} &= (1/\sqrt{2})\,\mathrm{e}^{\pm\mathrm{i}\alpha}[\mp\operatorname{cosech}\beta\,\partial_\alpha + \mathrm{i}\cosh\beta\,\partial_\beta \pm \coth\beta\,\partial_\gamma + (\mathrm{i}-\tfrac{1}{4}\eta)\sinh\beta]\\
\mathscr{U}_{0,\pm1} &= (1/\sqrt{2})\,\mathrm{e}^{\pm\mathrm{i}\gamma}[\mp\coth\beta\,\partial_\alpha - \mathrm{i}\cosh\beta\,\partial_\beta \pm \operatorname{cosech}\beta\,\partial_\gamma - (\mathrm{i}-\tfrac{1}{4}\eta)\sinh\beta]\\
\mathscr{U}_{0,0} &= \mathrm{i}\sinh\beta\,\partial_\beta + (\mathrm{i}-\tfrac{1}{4}\eta)\cosh\beta
\end{aligned} \tag{15}$$

where η is a real parameter. We note that

$$(\mathscr{U}_{\sigma,\tau})^\dagger = \mathscr{U}_{-\sigma,-\tau} \tag{16}$$

with respect to the measure $\sinh\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma$.

The full set of sl(4, ℝ) commutation relations, adapted to the chain sl(4, ℝ) ⊃ so(2, 2) is given by (8), (13), and their counterparts for $\mathscr{I}_0$ and $\mathscr{I}_\pm$, as well as by the following relation

$$\begin{aligned}
[\mathscr{U}_{\sigma\tau}, \mathscr{U}_{\sigma'\tau'}] &= (-1)^\tau\delta_{\tau,-\tau'}\sqrt{2}\langle 1\,\sigma, 1\,\sigma'|1\,\sigma+\sigma'\rangle\mathscr{I}_{\sigma+\sigma'}\\
&\quad + (-1)^\sigma\delta_{\sigma,-\sigma'}\sqrt{2}\langle 1\,\tau, 1\,\tau'|1\,\tau+\tau'\rangle\mathscr{I}_{\tau+\tau'}
\end{aligned} \tag{17}$$

resulting from (7), (14), and the corresponding relation for $[U_{\sigma\tau}, U_{\sigma'\tau'}]$. Here we have taken into account that the Wigner coefficients of the SU(1, 1) finite-dimensional non-unitary irreps are identical with those of SU(2) (Holman and Biedenharn 1966, Ui 1968, 1970), and that the tensor components of $\boldsymbol{J}$ and $\boldsymbol{\mathscr{I}}$ are J_0, $J_{\pm1} = \mp J_\pm/\sqrt{2}$, and $\mathscr{I}_0$, $\mathscr{I}_{\pm1} = \mathscr{I}_\pm/\sqrt{2}$, respectively.

The action of $\mathscr{U}_{\sigma\tau}$ on the complex conjugate Bargmann functions results from the Wigner–Eckart theorem with respect to su(1, 1)⊕su(1, 1), and is given by

$$\begin{aligned}
&\mathscr{U}_{\sigma\tau} V^{k*}_{m'm}(\alpha, \beta, \gamma)\\
&\quad = \sum_{k'} b_{k',k}\langle k\,m', 1\,\sigma|k'\,m'+\sigma\rangle_M\langle k\,m, 1\,\tau|k'\,m+\tau\rangle_M V^{k'*}_{m'+\sigma,m+\tau}(\alpha, \beta, \gamma).
\end{aligned} \tag{18}$$

Here the summation runs over $k' = k-1$, k, $k+1$, $b_{k',k}$ is some coefficient, and $\langle k\,m', 1\,\sigma | k'\,m'+\sigma\rangle_M$ denotes an SU(1, 1) Wigner coefficient coupling a unitary irrep D_k^+ with a non-unitary irrep with $K=1$ to get another unitary irrep $D_{k'}^+$ (Ui 1968, 1970). The coefficient $b_{k',k}$ can be easily found without calculation by an analytic continuation of (2). Ui (1968) has indeed shown that, apart from a phase arising from the two-valuedness of the square root, the SU(1, 1) Wigner coefficient $\langle k\,m', 1\,\sigma|k'\,m'+\sigma\rangle_M$ can be obtained from $\langle j\,m', 1\,\sigma|j'\,m'+\sigma\rangle$ by substituting k and k' for $-j$ and $-j'$, respectively. From the tabulated values of both Wigner coefficients, we obtain the relation

$$[(\langle j\,m, 1\,\sigma|j'\,m+\sigma\rangle)^2]_{k=-j,k'=-j'} = (-1)^{k-k'+\sigma}(\langle k\,m, 1\,\sigma|k'\,m+\sigma\rangle_M)^2. \tag{19}$$

Hence, direct comparison between equation (2), where $\sigma=\tau$ and $m'=m=-j$, and equation (18), where $\sigma=\tau$ and $m'=m=k$, leads to the relation

$$b_{k',k} = (-1)^{k'-k}[a_{j',j}]_{k=-j,k'=-j'}. \tag{20}$$

When combined with (3), the latter gives the results

$$b_{k-1,k} = \mathrm{i}(k-1)+\tfrac{1}{4}\eta \qquad b_{k,k} = -\tfrac{1}{4}\eta \qquad b_{k+1,k} = -\mathrm{i}k+\tfrac{1}{4}\eta. \tag{21}$$

We therefore conclude that under sl(4, ℝ) the set of (complex conjugate) Bargmann functions corresponding to the positive discrete series irreps separates into two subsets, corresponding to all integral or all half-integral values of k, respectively. Since, in the analytic continuation, the numerical values of the three sl(4, ℝ) independent Casimir operators remain unchanged, both subsets belong to the same sl(4, ℝ) irreps of the ladder series, $\mathfrak{D}^{\mathrm{ladd}}(0, 0; \eta)$ and $\mathfrak{D}^{\mathrm{ladd}}(\tfrac{1}{2}, \tfrac{1}{2}; \eta)$, as the corresponding subsets of Wigner functions.

The decomposition of these sl(4, ℝ) irreps into so(2, 2) irreps is, however, much more complicated than the corresponding decomposition into so(4) irreps. We indeed note that (18) is only valid for $k \geqslant \frac{3}{2}$. For $k=1$ or $\frac{1}{2}$, some of the Bargmann functions appearing on the right-hand side are not defined. However, by direct calculation, the following results can be proved:

$$\mathcal{U}_{-1,-1} V^{1*}_{1,1}(\alpha, \beta, \gamma) = \tfrac{1}{4}\eta \tag{22}$$

$$\mathcal{U}_{-1,-1} V^{1/2*}_{1/2,1/2}(\alpha, \beta, \gamma) = \tfrac{1}{2}(-\mathrm{i}+\tfrac{1}{2}\eta)\exp[-\tfrac{1}{2}\mathrm{i}(\alpha+\gamma)]\cosh\tfrac{1}{2}\beta. \tag{23}$$

On the right-hand side of (22), we recognise the identity representation, and on that of (23) the component $-\frac{1}{2}$, $-\frac{1}{2}$ of the two-dimensional non-unitary irrep with $K=\frac{1}{2}$. Hence the ladder series irreps of sl(4, ℝ) contain not only the positive discrete series irreps of su(1, 1)⊕su(1, 1), but also its finite-dimensional unitary and non-unitary irreps. This is not surprising since some of the sl(4, ℝ) generators, namely the operators $\mathcal{U}_{\sigma\tau}$, transform under a non-unitary irrep of su(1, 1)⊕su(1, 1). The basis states of the non-unitary su(1, 1)⊕su(1, 1) irreps, contained in the sl(4, ℝ) ladder series irreps, being unphysical, have no counterpart in the Hamiltonian spectra. Hence we shall not analyse the decomposition of the sl(4, ℝ) ladder series irreps any further, and we shall instead proceed to apply our results to the second Pöschl–Teller equation.

This equation is (Pöschl and Teller 1933)

$$\left[-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\hbar^2 a^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sinh^2 ax}-\frac{\lambda(\lambda+1)}{\cosh^2 ax}\right)-E_n\right]\psi_n(x)=0 \tag{24}$$

where a is some real parameter, the variable x runs over $[0, +\infty)$, κ, λ are two strength parameters, and $n \in \mathbb{N}$ labels the eigenvalues E_n and the wavefunctions $\psi_n(x)$. Since we are only interested in the negative-energy solutions of (24), we may assume $\lambda > \kappa > 1$. It is convenient to replace κ and λ by m' and m, defined by

$$\kappa = m' - m + \tfrac{1}{2} \qquad \lambda = m' + m - \tfrac{1}{2} \tag{25}$$

and to write the wavefunctions as $\psi_n^{(m',m)}(x)$. The condition $\lambda > \kappa > 1$ imposes the following restrictions on m' and m:

$$m' > m + \tfrac{1}{2} > 1. \tag{26}$$

Let us set

$$x = \beta/2a \qquad \beta \in [0, +\infty) \tag{27}$$

$$E_n = 2\hbar^2 a^2 \Lambda_n / M \tag{28}$$

and

$$\psi_n^{(m',m)}(x) = [(2k-1)a \sinh\beta]^{1/2} \varphi_n^{(m',m)}(\beta) \tag{29}$$

where k will be defined below in terms of m and n. Equation (24) is then transformed into the following equation

$$[d_{\beta\beta}^2 + \coth\beta\, d_\beta - (m'^2 + m^2 - 2m'm\cosh\beta)\operatorname{cosech}^2\beta + \Lambda_n + \tfrac{1}{4}]\varphi_n^{(m',m)}(\beta) = 0. \tag{30}$$

From (4), (11) and (12), it results that (30) coincides with the differential equation satisfied by the β-dependent part, $v_{m'm}^k(\beta)$, of the Bargmann SU(1, 1) functions corresponding to positive discrete series irreps, provided that $\Lambda_n = -(k - \tfrac{1}{2})^2$, where $k \in \{1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots\}$ and $m' - k$, $m - k \in \mathbb{N}$. From (25) and (26), these conditions imply that κ and λ must be half integral, and that

$$k = m - n \qquad n \in \mathbb{N}. \tag{31}$$

The eigenvalues can therefore be written in dimensionless units as

$$\begin{aligned} &\Lambda_n = -(m - n - \tfrac{1}{2})^2 = -\tfrac{1}{4}(\lambda - \kappa - 2n)^2 \\ &n = 0, 1, \ldots, [(\lambda - \kappa)/2] \end{aligned} \tag{32}$$

where $[(\lambda - \kappa)/2]$ denotes the largest integer contained in $(\lambda - \kappa)/2$. The corresponding normalised wavefunctions are given by (29), where

$$\varphi_n^{(m',m)}(\beta) = v_{m'm}^k(\beta). \tag{33}$$

By introducing an additional dependence on two auxiliary, angular variables α, $\gamma \in [0, 2\pi)$, the wavefunctions (29) are transformed into the extended wavefunctions

$$\begin{aligned} \Psi_n^{(m',m)}(x, \alpha, \gamma) &= (2\pi)^{-1} \exp(\mathrm{i}m'\alpha)\psi_n^{(m',m)}(x)\exp(\mathrm{i}m\gamma) \\ &= [(2k-1)a/4\pi^2]^{1/2} (\sinh\beta)^{1/2} V_{m'm}^{k*}(\alpha, \beta, \gamma) \end{aligned} \tag{34}$$

expressed in terms of the Bargmann functions (4).

Owing to (26), there is no one-to-one correspondence between the functions $v_{m'm}^k(\beta)$, m', $m = k, k+1, \ldots$, and the wavefunctions $\psi_n^{(m',m)}(x)$, nor between the functions $V_{m'm}^{k*}(\alpha, \beta, \gamma)$, m', $m = k$, $k+1, \ldots$, and the extended wavefunctions $\Psi_n^{(m',m)}(x, \alpha, \gamma)$. As a matter of fact, the functions $v_{m'm}^k(\beta)$ with $m > m' + \tfrac{1}{2}$ correspond to some replicas $\psi_n^{(m',m)}(x)$ of the true wavefunctions $\psi_n^{(m,m')}(x)$, $m > m' + \tfrac{1}{2}$, associated with the same potential of parameters $1 - \kappa$ and λ, since

$$\psi_n^{(m',m)}(x) = (-1)^{m'-m} \psi_n^{(m,m')}(x). \tag{35}$$

In addition, the functions $v^k_{mm}(\beta)$ correspond to some unphysical functions $\psi_n^{(m,m)}(x)$, i.e. functions not associated with a potential of the family.

It is now straightforward to obtain the sl(4, ℝ) dynamical potential algebra of the Pöschl–Teller potentials of the second kind. From (34), it follows that its generators, which we shall distinguish by a tilde, can be obtained from the corresponding ones for the Bargmann functions by a similarity transformation by $(\sinh\beta)^{1/2}$. The results are

$$\begin{aligned}
&\tilde{\mathscr{J}}_0 = -\mathrm{i}\,\partial_\alpha\\
&\tilde{\mathscr{J}}_\pm = \mathrm{e}^{\pm\mathrm{i}\alpha}[\mp(2a)^{-1}\partial_x - \mathrm{i}\coth 2ax\,\partial_\alpha + \mathrm{i}\,\mathrm{cosech}\,2ax\,\partial_\gamma \pm \tfrac{1}{2}\coth 2ax]\\
&\tilde{\mathscr{I}}_0 = -\mathrm{i}\partial_\gamma\\
&\tilde{\mathscr{I}}_\pm = \mathrm{e}^{\pm\mathrm{i}\gamma}[\pm(2a)^{-1}\partial_x - \mathrm{i}\,\mathrm{cosech}\,2ax\,\partial_\alpha + \mathrm{i}\coth 2ax\,\partial_\gamma \mp \tfrac{1}{2}\coth 2ax]\\
&\tilde{\mathscr{U}}_{\pm1,\pm1} = \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha+\gamma)}[-\mathrm{i}(2a)^{-1}\sinh 2ax\,\partial_x \mp \partial_\alpha \mp \partial_\gamma - \mathrm{i} + \tfrac{1}{4}\eta - \tfrac{1}{4}(2\mathrm{i}-\eta)\cosh 2ax] \qquad (36)\\
&\tilde{\mathscr{U}}_{\pm1,\mp1} = \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha-\gamma)}[-\mathrm{i}(2a)^{-1}\sinh 2ax\,\partial_x \pm \partial_\alpha \mp \partial_\gamma + \mathrm{i} - \tfrac{1}{4}\eta - \tfrac{1}{4}(2\mathrm{i}-\eta)\cosh 2ax]\\
&\tilde{\mathscr{U}}_{\pm1,0} = (1/\sqrt{2})e^{\pm\mathrm{i}\alpha}[\mathrm{i}(2a)^{-1}\cosh 2ax\,\partial_x \mp \mathrm{cosech}\,2ax\,\partial_\alpha \pm \coth 2ax\,\partial_\gamma - \tfrac{1}{2}\mathrm{i}\,\mathrm{cosech}\,2ax\\
&\qquad\qquad + \tfrac{1}{4}(2\mathrm{i}-\eta)\sinh 2ax]\\
&\tilde{\mathscr{U}}_{0,\pm1} = (1/\sqrt{2})\,\mathrm{e}^{\pm\mathrm{i}\gamma}[-\mathrm{i}(2a)^{-1}\cosh 2ax\,\partial_x \mp \coth 2ax\,\partial_\alpha \pm \mathrm{cosech}\,2ax\,\partial_\gamma + \tfrac{1}{2}\mathrm{i}\,\mathrm{cosech}\,2ax\\
&\qquad\qquad - \tfrac{1}{4}(2\mathrm{i}-\eta)\sinh 2ax]\\
&\tilde{\mathscr{U}}_{0,0} = \mathrm{i}(2a)^{-1}\sinh 2ax\,\partial_x + \tfrac{1}{4}(2\mathrm{i}-\eta)\cosh 2ax.
\end{aligned}$$

One should remark that they could also have been obtained by analytic continuation from the generators of the dynamical potential algebra of the first Pöschl–Teller potential family.

From (12), (18), (31) and (34), the action of the sl(4, ℝ) generators on the extended wavefunctions is given by

$$\begin{aligned}
&\tilde{\mathscr{J}}_0\Psi_n^{(m',m)}(x,\alpha,\gamma) = m'\Psi_n^{(m',m)}(x,\alpha,\gamma)\\
&\tilde{\mathscr{J}}_+\Psi_n^{(m',m)}(x,\alpha,\gamma) = [(m'-m+n+1)(m'+m-n)]^{1/2}\Psi_n^{(m'+1,m)}(x,\alpha,\gamma)\\
&\tilde{\mathscr{I}}_0\Psi_n^{(m',m)}(x,\alpha,\gamma) = m\Psi_n^{(m',m)}(x,\alpha,\gamma),\\
&\tilde{\mathscr{I}}_+\Psi_n^{(m',m)}(x,\alpha,\gamma) = [(n+1)(2m-n)]^{1/2}\Psi_{n+1}^{(m',m+1)}(x,\alpha,\gamma)
\end{aligned} \qquad (37)$$

and

$$\begin{aligned}
\tilde{\mathscr{U}}_{\sigma\tau}\Psi_n^{(m',m)}&(x,\alpha,\gamma)\\
&= \sum_{n'=n+\tau-1}^{n+\tau+1} d_{n'}(m-n)\langle m-n\; m', 1\;\sigma | m-n'+\tau\; m'+\sigma\rangle_M\\
&\quad\times\langle m-n\; m, 1\;\tau | m-n'+\tau\; m+\tau\rangle_M \Psi_{n'}^{(m'+\sigma,m+\tau)}(x,\alpha,\gamma)
\end{aligned} \qquad (38)$$

where

$$d_{n'}(m-n) = \begin{cases} [-\mathrm{i}(m-n)+\tfrac{1}{4}\eta][(2m-2n-1)/(2m-2n+1)]^{1/2} \\ \qquad\qquad \text{if } n'=n+\tau-1 \\ -\tfrac{1}{4}\eta \qquad\quad \text{if } n'=n+\tau \\ [\mathrm{i}(m-n-1)+\tfrac{1}{4}\eta][(2m-2n-1)/(2m-2n-3)]^{1/2} \\ \qquad\qquad \text{if } n'=n+\tau+1. \end{cases} \qquad (39)$$

As already proved by other authors (Frank and Wolf 1985, Barut *et al* 1987b), the generators of so(2, 2) connect together the eigenstates associated with the same eigenvalue Λ_n, given by (32), but with different potentials corresponding to the sets of quantised potential strengths (m', m), $(m'\pm 1, m)$, and $(m', m\pm 1)$. All such states belong to a single su(1, 1)⊕su(1, 1) irrep $D_k^+ \oplus D_k^+$. After substituting $-\mathrm{i}\partial_\alpha$ and $-\mathrm{i}\partial_\gamma$ for m' and m respectively, the Pöschl–Teller Hamiltonian H, as defined in (24), is essentially the su(1, 1)⊕su(1, 1) Casimir operator, since

$$\tilde{\mathscr{J}}^2 = \tilde{\mathscr{I}}^2 = -M(2\hbar^2 a^2)^{-1} H - \tfrac{1}{4}. \tag{40}$$

In addition, the generators $\tilde{\mathscr{U}}_{\sigma\tau}$ of sl(4, ℝ) can connect eigenstates associated with different eigenvalues. In particular, $\tilde{\mathscr{U}}_{00}$ generates transitions between eigenstates corresponding to the same potential and values of n differing by one unit. All the eigenstates of the family of Pöschl–Teller potentials with half-integral values of κ and λ, such that $\kappa+\lambda$ is even (odd) and $\lambda-\kappa$ is odd (even), belong to the carrier space of $\mathfrak{D}^{\mathrm{ladd}}(0, 0; \eta)[\mathfrak{D}^{\mathrm{ladd}}(\frac{1}{2}, \frac{1}{2}; \eta)]$. However, such carrier spaces also contain extra copies of the potential family eigenstates, as well as some unphysical states.

The second Pöschl–Teller equation also has non-negative energy solutions, that have not been discussed in the present paper. The zero-energy solution can be expressed in terms of the Bargmann function $V^{1/2*}_{m'm}(\alpha, \beta, \gamma)$ corresponding to the positive discrete series irrep $D^+_{1/2}$, while the positive-energy ones are given in terms of the continuous principal series irreps of SU(1, 1), C^0_q and $C^{1/2}_q$ (Barut *et al* 1987b). In principle, they could be analysed along sl(4, ℝ) lines as the discrete states. However, as far as the author knows, the SU(1, 1) Wigner coefficients coupling a continous principal series irrep with a finite-dimensional non-unitary irrep have not been determined so far.

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